Math 115A A, Lecture 2
Real Analysis

Sample Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

Decide whether each set V with the laws of addition and scalar multiplication given is a vector space, and prove your answer.

(a) [5pts.] $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$ with $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ and

$$c(a_1, a_2) = \begin{cases} (0, 0) \text{ if } c = 0 \\ \left(ca_1, \frac{a_2}{c}\right) \text{ if } c \neq 0 \end{cases}$$

Solution: No, V is not a vector space. Notice that if $c, d \in \mathbb{R}$ and $(a_1, a_2) \in V$, then $(c+d)(a_1, a_2) = ((c+d)a_1, \frac{a_2}{c+d})$, but $c(a_1, a_2)+d(a_1, a_2) = ((c+d)a_1, \frac{a_2}{c}+\frac{a_2}{d})$. Since $\frac{1}{c+d} \neq \frac{1}{c} + \frac{1}{d}$ in general, this violates axiom (VS8).

(b) [5pts.] V is the set of even functions $f : \mathbb{R} \to \mathbb{R}$. (An even function is one for which $f(t) = f(-t)$.)

Solution: V is a vector space; in fact, we claim it is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$, the vector space of functions from $\mathbb{R}$ to itself. For we can check:

- The additive identity in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is the function h such that $h(t) = 0$ for all t. This is certainly even, since $h(t) = 0 = h(-t)$ for all t.
- If f and g are even, then $(f + g)(t) = f(t) + g(t) = f(-t) + g(-t) = (f + g)(-t)$, so the function $f + g$ is also even.
- If f is even and $c \in \mathbb{R}$, then $(cf)(t) = c \cdot f(t) = c \cdot f(-t) = (cf)(-t)$, so the function cf is also even.

Problem 2.

Let $S \subset V$ be a subset of a real vector space V.

(a) [5pts.] What does it mean to say that S is linearly independent?

Solution: We say that S is linearly independent if there is no finite set of vectors $u_1, \cdots, u_n \in S$ and scalars $a_1, \cdots, a_n \in \mathbb{R}$ not all zero such that $a_1u_1 + \cdots a_nu_n = 0$.

(b) [5pts.] Prove that if S is linearly independent, and $v \notin S$, then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Solution: First suppose that $S \cup \{v\}$ is linearly dependent. Then there is a nontrivial representation of zero $a_1u_1 + \cdots + a_nu_n = 0$ for vectors $u_1, \cdots, u_n \in S$ and scalars $a_1, \cdots, a_n \in \mathbb{R}$ not all zero. Since S was linearly independent, one of these vectors, say u_1, must in fact be v, and its coefficienta_1 must be nonzero.

So we can rewrite $0 = a_1 v + a_2 u_2 + \cdots + a_n u_n$, or $v = \frac{-a_2}{a_1} u_2 + \cdots + \frac{-a_n}{a_1} u_n$. We conclude that $v \in \text{span}(S)$. Conversely, suppose $v \in \text{span}(S)$. Then there are some $u_1, \cdots, u_n \in S$ and $a_1, \cdots, a_n \in \mathbb{R}$ such that $v = a_1 u_1 + \cdots a_n u_n$, which we can rewrite as $0 = a_1 u_1 + \cdots + a_n u_n - v$. Since v is not any of the u_n, this is a nontrivial representation of zero among the elements of $S \cup \{v\}$. Ergo $S \cup \{v\}$ is linearly dependent.

Problem 3.

Let $S \subset P_3(\mathbb{R})$ be the set $\{x^2 - 3x + 2, x^3 - 1, x^2 - 1, x^2 + 2x - 3\}$.

(a) [5pts.] Is S linearly independent or dependent?

Solution: S is linearly dependent. For if we set $a(x^2 - x + 2) + b(x^3 - 1) + c(x^2 - 1) + d(x^2 + 2x - 3) = 0$, we obtain the equations

$$\begin{cases} b = 0 \\ a + c + d = 0 \\ 2d - 3a = 0 \\ 2a - b - c - 3d = 0 \end{cases}$$

Solving these shows that $a = 2, b = 0, c = -5, d = 3$ is a solution, and gives a nontrivial representation of 0 as a linear combination of elements of S.

(b) [5pts.] What is span(S)? [Hint: Consider factoring.]

Solution: Notice that the four polynomials above factor as $(x - 1)(x - 2)$, $(x-1)(x^2+x+1)$, $(x-1)(x+1)$, and $(x-1)(x+3)$. We claim that $\text{span}(S) = \{p(x) \in P_3(\mathbb{R}) : p(1) = 0\}$. To verify this, notice that an arbitrary polynomial in this subspace can be written $(x - 1)(ax^2 + bx + c)$, and is therefore a linear combination of elements of S as follows:

$$a(x - 1)(x^2 + x + 1) + (2b - a - c)(x - 1)(x + 1) + (c - b)(x - 1)(x + 2).$$

Problem 4.

Let V be the subset of $\text{Mat}_{3x3}(\mathbb{R})$ consisting of matrices of trace zero. (Recall that the trace of a matrix is the sum of its diagonal entries, so $\text{tr}(A) = A_{11} + A_{22} + A_{33}$.)

(a) [5pts.] Prove that V is a subspace of $\text{Mat}_{3\times3}(\mathbb{R})$.

Solution: First, observe that the zero 3×3 matrix has trace zero (the sum of its diagonal entries is certainly zero), so $0 \in V$. Furthermore, if $A, B \in V$, then

$\mathrm{tr}(A+B) = (A+B)_{11}+(A+B)_{22}+(A+B)_{33} = A_{11}+B_{11}+A_{22}+B_{22}+A_{33}+B_{33} = \mathrm{tr}(A)+\mathrm{tr}(B)$. So V is closed under addition. Finally, if $c \in \mathbb{R}$ and $A \in V$, then $\mathrm{tr}(cA) = (cA)_{11} + (cA)_{22} + (cA)_{33} = cA_{11} + cA_{22} + cA_{33} = c(\mathrm{tr}(A))$. Ergo V is also closed under scalar multiplication.

(b) [5pts.] Determine the dimension of V, and prove your answer.

Solution: Any matrix A in V may be expressed as

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix}$$

We claim there is a basis β for V consisting of eight elements, the first six of which are the matrices E^{ij} for $i \neq j$. (Recall that the matrix E^{ij} is the matrix for which $E_{ij} = 1$ and all other entries are zero.) The remaining two entries in the basis are the matrix F which has $F_{11} = 1$, $F_{33} = -1$, and all other entries zero, and the matrix G which has $G_{22} = 1$, $G_{33} = -1$, and all other entries zero. Then the arbitrary matrix in V above is a linear combination of elements of β via $A = aF + eG + bE^{12} + cE^{13} + dE^{21} + fE^{23} + gE^{31} + hE^{32}$. So β spans V, and furthermore an arbitrary linear combination of the eight elements of β (with the same coefficients, say) is equal to zero only if $a = b = c = d = e = f = g = h = 0$, so β is a basis as claimed. Therefore $\dim(V) = 8$.

Problem 5.

Let W_1 and W_2 be subspaces of a vector space V such that W_1 has dimension n and W_2 has dimension m, and $m \geq n$.

(a) [5pts.] Prove that $\dim(W_1 \cap W_2) \leq n$. [Hint: Produce a linearly independent set in W_1.]

Solution: Let β be a basis for $W_1 \cap W_2$. Then in particular β is a linearly independent subset of W_1, so the number of elements in β is less than or equal to the dimension of W_1, namely n. Therefore $\dim(W_1 \cap W_2) \leq n$.

(b) [5pts.] Prove that $\dim(W_1 + W_2) \leq m + n$. [Hint: Produce a generating set in $W_1 + W_2$.]

Solution: Recall from the homework that $W_1 \subset W_1 + W_2$ and $W_2 \subset W_1 + W_2$. Choose a basis β for W_1, containing n elements, and a basis ϵ for W_2 containing m elements. We claim that $G = \beta \cup \epsilon$ spans $W_1 + W_2$. For if $w_1 + w_2$ is an arbitary element of $W_1 + W_2$, w_1 is a linear combination of elements of β and w_2 is a linear combination of elements of W_2, so $w_1 + w_2$ is a linear combination elements of G. Ergo G is a generating set for $W_1 + W_2$ with $n + m$ elements, which implies that the dimension of $W_1 + W_2$ is less than or equal to $n + m$.